

Questions of Stability near Black Hole Critical Points

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Abstract

In this letter, we discuss how thermal fluctuations can effect the stability of (generally) charged black holes when close to certain critical points. Our novel treatment utilizes the black hole area spectrum (which is, for definiteness, taken to be evenly spaced) and makes an important distinction between fixed and fluctuating charge systems (with these being modeled, respectively, as a canonical and grand canonical ensemble.) The discussion begins with a summary of a recent technical paper [1]. We then go on to consider the issue of stability when the system approaches the critical points of interest. These include the d -dimensional analogue of the Hawking-Page phase transition, a phase transition that is relevant to Reissner-Nordstrom black holes and various extremal-limiting cases.

It has often been suggested that any viable theory of quantum gravity must somehow account for black hole thermodynamics [2]. In particular, a prospective fundamental theory would be expected to provide a statistical explanation for the Bekenstein-Hawking entropy [3,4]: $S_{BH} = \frac{1}{4G}A$, where A is the surface area of the black hole horizon.¹

An important subplot is the leading-order quantum correction to this classical area law. Indeed, there has been substantial interest in deducing this correction, with many different techniques having been called upon for just this purpose (see [1] for a thorough list of references). A feature that is common to all of the relevant studies is a leading-order correction that arises at the logarithmic order in S_{BH} . Nonetheless, there is significant disagreement on the value of the proportionality constant or logarithmic *prefactor*.

This conflict over the prefactor can be partially resolved when one considers an often overlooked point: there are actually two distinct sources for this logarithmic correction. More precisely, one can anticipate both an uncertainty in the number of microstates describing a

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¹Here and throughout, all fundamental constants - besides the gravitational coupling constant, G - are set to unity.

black hole with a fixed geometry *and* a quantum correction due to thermal fluctuations in the area of the horizon. In principle, these two contributions should be completely separable, at least to the logarithmic order [5,6]. For the remainder, we will focus our attention on the latter (thermal) correction.

It has sometimes been implied that the thermal correction can be obtained, given a suitable canonical framework, without any reference to the fundamental theory of quantum gravity. (This clearly can not be the case for the other, microcanonical correction, which ultimately must refer to the underlying degrees of freedom.) However, in a recent paper [1], we have argued that an important element of the quantum theory - namely, the black hole area spectrum - must inevitably be accounted for in any such calculation. With this realization in mind, we proceeded to calculate the thermal correction to the black hole entropy; both generically and for an assortment of special models (in particular, various limiting cases of a Reissner-Nordstrom black hole in an anti-de Sitter, or AdS, spacetime of arbitrary dimensionality).

For the sake of definiteness, we made the decision in [1] to work with an evenly spaced area spectrum (and will continue to do so here). Although this spectral form remains somewhat controversial, let us take note of the support in the literature; beginning with the heuristic arguments of Bekenstein [7] and more rigorous treatments since (*e.g.*, [8–12]).²

Another novelty of our prior work [1] is that a crucial distinction was made between black holes with a fixed charge and those for which the charge is allowed to fluctuate. (The charge can legitimately be regarded as a fixed quantity only under certain circumstances; for instance, a black hole confined within a neutral heat bath.) Accordingly, we modeled the system as both a canonical ensemble and a *grand* canonical ensemble (respectively). The formal differences will be further clarified below.

The remainder of the letter is organized as follows. We first give a summarized account of the preceding work [1], as this establishes the foundation for the subsequent analysis. Next, we investigate the issue of black hole stability when the system approaches certain critical points (discussed thoroughly below). The letter ends with a brief conclusion.

In review of [1], let us first consider the case of a black hole with a fixed charge. Given the premise of a black hole in a state of thermal equilibrium (*i.e.*, a black hole in a “box”), an appropriate model for the system is a canonical ensemble of particles and fields. Under the assumption of an evenly spaced area spectrum, $A(n) \sim n$ ($n = 0, 1, 2, \dots$), and a semi-classical (or large black hole) regime, it follows that the canonical partition function can be expressed as

$$\mathcal{Z}_C(\beta) = \int_0^\infty dn \exp(-\beta E(n) + \epsilon n) . \quad (1)$$

Here, β^{-1} is the fixed temperature and $E(n)$ is the energy of the n -th level. Furthermore, the degeneracy has been fixed by way of the entropic area law; that is, $S(n) = \epsilon n$, with ϵ being a dimensionless, positive parameter of the order unity.

²It is also noteworthy that, even in the context of loop quantum gravity, certain authors have advocated for an equally spaced spectrum [13].

Evaluating the above and then employing text-book thermodynamics, we found the following for the canonical entropy:

$$S_C \approx S_{BH} - \frac{1}{2} \ln \left[\frac{E_0''}{E_0'} \right] , \quad (2)$$

where a prime (always) denotes differentiation with respect to n , a subscript of 0 indicates a quantity evaluated at $n = n_0 \equiv < n >$, and the classical black hole entropy has been identified as $S_{BH} = \epsilon n_0$. (An approximation sign always signifies that irrelevant constant terms and higher-order corrections have been neglected.)

Take note of the right-most term, which is the anticipated logarithmic correction to the classical area law. As an immediate consequence of the formalism, the argument of this logarithm must be strictly positive. Given that the temperature can not be negative (by virtue of cosmic censorship) and that $E_0' = \epsilon \beta^{-1}$ (via the first law of thermodynamics), one is able to identify a pair of *stability* conditions: $E_0' > 0$ and $E_0'' > 0$.

The canonical partition function can also be used to evaluate the variation of the spectral number. For this calculation, we have obtained

$$(\Delta n)^2 \equiv < n^2 > - < n >^2 \approx \frac{E_0'}{\epsilon E_0''} . \quad (3)$$

Keep in mind that this variation serves as a direct measure of the thermal fluctuations in the area or entropy; that is, $\Delta S_{BH} \sim \Delta A \sim \Delta n$. Consequently, Eq.(2) can be elegantly rewritten as

$$S_C \approx S_{BH} + \ln[\Delta S_{BH}] . \quad (4)$$

Secondly, let us consider the case of a black hole with a fluctuating charge. There should now be an additional quantum number that accounts for this dynamical charge and it is, therefore, appropriate to model the system as a grand canonical ensemble. Given a uniformly spaced area spectrum and the independence of the quantum numbers, we were able to deduce the following spectral form:

$$A(n, m) \sim n + \alpha m^p , \quad n, |m| = 0, 1, 2, \dots , \quad (5)$$

where the “new” quantum number, m , directly measures the black hole charge according to $Q = me$ (with e being a fundamental unit of electrostatic charge). Also, α is a positive constant and p is a positive, rational number.

We are now in a position to write down the grand canonical partition function,

$$\mathcal{Z}_G(\beta, \lambda) = \int_{-\infty}^{\infty} dm \int_0^{\infty} dn \exp(-\beta E(n, m) + \epsilon(n + \alpha m^p) + \lambda m) , \quad (6)$$

where λ is a chemical (or electric) potential and ϵ is (again) a dimensionless, positive parameter.

After some lengthy evaluation, the *grand* canonical entropy was eventually found to be as follows:

$$S_G \approx S_{BH} - \frac{1}{2} \ln \left[\frac{\Psi_0}{(E_0')^2} \right] , \quad (7)$$

where we have defined

$$\Psi_0 \equiv E_0'' \left(\ddot{E}_0 - \alpha p(p-1)m_0^{p-2}E_0' \right) - (\dot{E}_0')^2 \quad (8)$$

and identified

$$S_{BH} = \epsilon \langle n + \alpha m^p \rangle. \quad (9)$$

Furthermore, a subscript of 0 now indicates a quantity evaluated at *both* $n = n_0$ and $m = m_0 \equiv \langle m \rangle$, and a dot represents a differentiation with respect to m .

In view of the revised form of the logarithmic correction (7), the relevant stability conditions now become $E_0' > 0$ and $\Psi_0 > 0$. (Note that the relation $E_0' = \epsilon\beta^{-1}$ is still valid.)

As for the prior (canonical) treatment, the grand canonical partition function can be used to evaluate the variations in the quantum numbers:

$$(\Delta n)^2 \approx \frac{E_0' [\Psi_0 + (\dot{E}_0')^2]}{\epsilon E_0'' \Psi_0}, \quad (10)$$

$$(\Delta m)^2 \approx \frac{E_0' E_0''}{\epsilon \Psi_0}. \quad (11)$$

In the work of interest [1], we also translated our generic formalism into the framework of a d -dimensional ($d \geq 4$) AdS-Reissner-Nordstrom black hole. Beginning with the defining relation for the horizon [14] and the first law of black hole mechanics, we could readily obtain the spectral form of the energy. In this regard, it is useful to note that $R(n, m) \sim [A(n, m)]^{\frac{1}{d-2}}$ where R is the radius of the horizon and m may or may not be fixed.

For the case of a fixed charge, one finds that

$$E(n) = \frac{1}{\omega_d} \left[\frac{\mathcal{L}^{d-1}}{L^2} n^{\frac{d-1}{d-2}} + \mathcal{L}^{d-3} n^{\frac{d-3}{d-2}} + \frac{\omega_d^2 Q^2}{\mathcal{L}^{d-3}} n^{-\frac{d-3}{d-2}} \right], \quad (12)$$

where L is the AdS curvature parameter, ω_d is a scaled form of the d -dimensional Newton constant, and \mathcal{L} is a length scale such that $R_0^{d-2} = \mathcal{L}^{d-2} S_{BH}/\epsilon$. It should be kept in mind that the curvature parameter, L , can always be viewed as the effective box size (*i.e.*, the spatial extent) of the system.

Meanwhile, for the fluctuating-charge scenario, the spectral energy can be expressed as follows:

$$E(n, m) = \frac{1}{\omega_d} \left[\frac{\mathcal{L}^{d-1}}{L^2} \mathcal{A}^{\frac{d-1}{d-2}} + \mathcal{L}^{d-3} \mathcal{A}^{\frac{d-3}{d-2}} + \frac{\omega_d^2 e^2 m^2}{\mathcal{L}^{d-3}} \mathcal{A}^{-\frac{d-3}{d-2}} \right], \quad (13)$$

where $\mathcal{A} \equiv (n + \alpha m^p)$ is the dimensionless area. Take note of the implicit appearance of the parameter p in the above expression. For some specific cases, it is possible to fix p by an inspection of the classical black hole area, as a function of charge, at extremality. (In particular, $p = 1$ if $R \gg L$ and $p = \frac{d-2}{d-3}$ if $R \ll L$ [1].) It is, however, an interesting feature of this Reissner-Nordstrom model that most of the results do not depend *explicitly* on p . This oddity follows from a calculation of Ψ_0 (8), which shows that, if we write $E(\mathcal{A}) = f(\mathcal{A}) + m^2 g(\mathcal{A})$, then $\Psi_0 = 2g_0 f_0'' + 2m_0^2 g_0 g_0'' - 4m_0^2 (g_0')^2$.

Given the above expressions, it becomes a straightforward process to determine the thermal correction to the entropy (as well as the thermal fluctuations) for any AdS-Reissner-Nordstrom black hole. In the prior work, we considered a number of interesting limits, where the calculations are most easily interpreted. Before summarizing these findings, let us point out that, in all cases considered, the canonical (grand canonical) entropy takes on a particularly simple form,

$$S_{C(G)} = S_{BH} + b \ln[S_{BH}] , \quad (14)$$

where b is a non-negative, rational number that is independent of the dimensionality of the spacetime. Except for the independence of d , we expect that this form persists under more general circumstances.

Let us now recall, case by case, some of the more prominent outcomes of [1].

(i) $L \ll R$ and $Q \sim 0$: This case can be viewed as a neutral black hole in a small box. When the charge is fixed, the logarithmic prefactor is $b = 1/2$;³ whereas $b = 1$ if the charge is allowed to fluctuate. This implies that each quantum number - that is, each freely fluctuating parameter - induces a thermal correction of precisely $\frac{1}{2} \ln S_{BH}$. (Let us emphasize that, for a fluctuating-charge scenario, the condition $Q \sim 0$ should really be written as $\langle Q \rangle \sim 0$. Regardless of this neutrality, the fluctuations can still be, and generally are, quite large.) As for the variations in the quantum numbers, we found that, for a fixed charge, $\Delta n \sim n_0^{\frac{1}{2}}$ and, when the charge is fluctuating, $\Delta n, \Delta m \sim \mathcal{A}_0^{\frac{1}{2}}$. Since $p = 1$ in this limit (see above), it follows that, for both scenarios, $\Delta S_{BH} \sim S_{BH}^{\frac{1}{2}}$.

(ii) $L \sim R$ and $Q \sim 0$: It turns out that, for a sufficiently large L and vanishing charge, the stability condition $E_0'' > 0$ (or the analogue, $\Psi_0 > 0$) will be violated. The maximal value of box size (defined by the saturation of the stability condition),

$$L_{max} \equiv \sqrt{\frac{d-1}{d-3}} R_0 = \sqrt{\frac{d-1}{d-3}} \mathcal{L} n_0^{\frac{1}{d-2}} , \quad (15)$$

can be identified as the d -dimensional analogue of the Hawking-Page phase-transition point [16]. By way of a perturbative expansion, we have shown that, near the transition point, $b = 1$. Also, $\Delta n \sim n_0$ when the charge is fixed and $\Delta n \sim \mathcal{A}_0$ for the dynamical-charge scenario. Moreover, it can be readily verified that, near this point, the charge fluctuations are always suppressed: $\Delta m \sim \text{constant}$. However, the rather large fluctuations in the quantum number n (and, hence, in the area) are indicative of an instability in the system. This notion will be elaborated on below.

(iii) $L \ll R$ and $Q \sim Q_{ext}$: Here, we are using Q_{ext} to denote the charge of an extremal black hole. Significantly, the extremal limit is also the limit of vanishing temperature (*i.e.*, $\beta^{-1} = 0$) and, therefore, signifies the saturation of the stability condition $E_0' > 0$. Hence, $|Q_{ext}|$ serves as an upper bound on the magnitude of the charge. Using a perturbative expansion, we found that, close to extremality, the logarithmic prefactor vanishes ($b = 0$)

³As a point of interest, the exact same value, $b = 1/2$, was found for the (3-dimensional) BTZ black hole model [15], irrespective of the curvature parameter.

and all fluctuations are completely suppressed ($\Delta n, \Delta m \sim \text{constant}$). This suppression can be viewed as a natural mechanism for enforcing cosmic censorship in a near-extremal regime.

(iv) $L \gg R$ and $Q \sim Q_{ext}$ and fixed charge: First note that, in this regime of large box size (or, equivalently, the asymptotically flat space limit), the fixed and fluctuating charge scenarios turn out to be qualitatively much different, and so these will be dealt with as separate cases. In the current (fixed-charge) case, the thermal fluctuations and logarithmic prefactor were shown, once again, to be completely suppressed when the black hole is near extremality.

(v) $L \gg R$ and $Q \sim Q_{min}$ and fixed charge: By virtue of the stability condition $E'_0 > 0$, it can be demonstrated that (when L is large) there is also a *lower* bound on the charge. Quantitatively, one finds that

$$Q_{min}^2 \equiv \frac{R^{2(d-3)}}{(2d-5)\omega_d^2} = \frac{\mathcal{L}^{2(d-3)}}{(2d-5)\omega_d^2} n_0^{\frac{2(d-3)}{d-2}}, \quad (16)$$

and note that $Q_{min}^2 = \frac{1}{2d-5} Q_{ext}^2$.⁴ This point of minimal charge, but finite temperature, is certainly indicative of a phase transition. (The existence of such a phase transition for Reissner-Nordstrom black holes was first established by Davies [17].) In fact, close to the minimal charge, the ensemble behaves in precisely the same way as it does near the Hawking-Page transition point (*cf.* case *ii*). That is, the logarithmic prefactor is quite large, $b = 1$, and the area fluctuates according to $\Delta n \sim n_0$. Again, such behavior is suggestive of an instability in the system near the critical point.

(vi) $L \gg R$ and dynamical charge: For this case, we have shown that it is impossible to simultaneously satisfy both of the relevant stability conditions, $E'_0 > 0$ and $\Psi_0 > 0$. In the language of the prior case, this instability is a consequence of the coincidence $Q_{ext}^2 = Q_{min}^2$. Hence, when the charge is allowed to fluctuate, a non-extremal black hole in a large box can *not* possibly be stable.

We would now like to extend the prior analysis by asking (and then answering) the following questions: just how close can the system approach the extremal limit (*cf.* cases *iii* and *iv*) and the phase-transition points (*cf.* cases *ii* and *v*) before instability ensues? With regard to the former, given the suppression of the thermal fluctuations, one might naively expect that a near-extremal black hole can come infinitesimally close to its point of extremality. (Stability near the phase-transition points is somewhat less clear, inasmuch as these thermal fluctuations are far from suppressed.) However, we will argue below that this, in fact, can *not* be the case.⁵

To help answer these questions, let us work under the following sensible premise: stability necessitates that the thermal fluctuations can *not* be any larger than the *deviation* from the

⁴Although Q_{ext}^2 and Q_{min}^2 are relatively close in units of black hole charge, it should be remembered that, for a semiclassical black hole, Q_{ext}^2 is a very large quantity in Planck units, and so cases *iv* and *v* can safely be viewed as isolated regimes.

⁵Note that our formalism does break down at, precisely, any of the critical points under consideration. Hence, we can not comment on the nature of a *perfectly* extremal black hole, assuming that such an entity can even exist. For discussion and references on this controversial issue, see [18].

relevant critical point. We will show how this notion can be put in more rigorous terms upon examining some specific cases.

Let us proceed by first considering case *ii* for a (neutral) black hole near the Hawking-Page phase-transition point. For simplicity, we will, for the time being, focus on the fixed-charge scenario. Recall that the phase transition takes place when L reaches a maximal value. It is, however, more appropriate to keep L as a fixed parameter and consider changes in the dynamical parameter n . In this spirit, Eq.(15) can be re-expressed as a minimal (or critical) bound, n_c , on the expectation value of n . That is,

$$n_0 > n_c \equiv \left[\sqrt{\frac{d-3}{d-1}} \frac{L}{\mathcal{L}} \right]^{d-2}. \quad (17)$$

Since the system is supposed to be in the “proximity” of this transition point, it is appropriate to write

$$n_0 = n_c + \varepsilon, \quad \text{where} \quad \varepsilon \equiv n_c^\gamma \delta, \quad (18)$$

such that $\gamma < 1$ and δ is a constant, positive parameter of the order unity. Note that ε represents the “deviation” that was alluded to above.

Given the above expansion (and keeping in mind that $Q \sim 0$), some straightforward calculation shows that, to lowest order in ε , $E'_0 \sim n_c^{-\frac{1}{d-2}}$, $E''_0 \sim n_c^{-\frac{2d-3}{d-2}} \varepsilon$, and so

$$\Delta n \approx \sqrt{\frac{E'_0}{E''_0}} \sim \frac{n_c}{\sqrt{\varepsilon}}. \quad (19)$$

Now let us reconsider our *ansatz* for stability. In quantitative terms, this essentially means that $\Delta n < \varepsilon$, which translates into $n_c < \varepsilon^{\frac{3}{2}}$ or, equivalently, $\gamma > 2/3$ (*cf.* Eq.(18)). For a macroscopically large black hole (*i.e.*, n_0 or $n_c \gg 1$), $n_c^{2/3} \ll n_c$, so that the deviation (ε) can be regarded as small relative to the size of the black hole. Nonetheless, the minimum deviation is still a very large number in Planck units and establishes our previous claim regarding instability near the Hawking-Page phase-transition point. As a further point of interest, one obtains $b = 2/3$ for the logarithmic prefactor when the system is “teetering” on instability.

One can apply the same general technique to the fluctuating-charge version of case *ii* and obtain very similar results. (It is not necessary to fix the parameter p , so long as $p \geq 1$. Also note that, for completeness, one should compare both Δn and Δm with ε .) It is interesting to note that $\Delta m \sim \varepsilon$, so that both Δn and Δm go as $n_c^{2/3}$ near the point where instability ensues.

One can also apply this methodology to the Reissner-Nordstrom phase transition of case *v*. To recall, the stability condition $E''_0 > 0$ necessitates that, for a black hole in a large box, there is a minimum value of *fixed* charge and, hence, a maximum value of n_0 (*cf.* Eq.(16)). In this case, the same approach as above leads to *precisely* the same constraint on the deviation (ε); that is, $\gamma > 2/3$, where $\varepsilon \sim n_c^\gamma$ and n_c is now the critical point relevant to case *v*. To the best of our knowledge, this is the first mention of such a similarity between two otherwise unrelated phase transitions [16,17].

Let us next ponder the question of stability for near-extremal black holes. For illustrative purposes, we will specifically concentrate on case *iv* (large L and fixed Q); nonetheless, the same features can be shown to persist for *any* of the near-extremal models of interest.⁶

Given that $L \gg R$ (and so the first term in Eq.(12) can safely be disregarded), it is not difficult to solve for the extremal value of charge and then re-express this as a critical minimum, n_c , on the expectation value of n . Following this procedure, we obtain

$$n_0 > n_c \equiv \left[\frac{\omega_d |Q|}{\mathcal{L}^{d-3}} \right]^{\frac{d-2}{d-3}}. \quad (20)$$

As before, we can appropriately write

$$n_0 = n_c + \varepsilon, \quad \text{where} \quad \varepsilon \equiv n_c^\gamma \delta, \quad (21)$$

with the understanding that $\gamma < 1$ and $\delta \sim \mathcal{O}[1]$.

Utilizing the above expansion (and remembering that $L \gg R$), we can deduce, to lowest order in ε , $E'_0 \sim n_c^{-\frac{d-1}{d-2}}$, $E''_0 \sim n_c^{-\frac{d-1}{d-2}} \varepsilon$, so that

$$\Delta n \approx \sqrt{\frac{E'_0}{E''_0}} \sim \sqrt{\varepsilon}. \quad (22)$$

It is clear that the thermal fluctuation, Δn , goes to zero as $\varepsilon \rightarrow 0$. Hence, we can now set $\gamma = 0$ without endangering stability, and so $\varepsilon = \delta$. The pertinent question becomes is there any lower limit on δ ? In fact, by imposing our stability constraint, $\Delta n < \varepsilon$, we must have $\delta > 1$. This means that there is, indeed, a natural limit on just how close a near-extremal black hole can approach absolute extremality; roughly (but not less than) one Planck unit of area. It is reasonable to suggest that this censoring mechanism is a manifestation of some *generalized* form of the third law of thermodynamics. Let us re-emphasize that the same censorship occurs for *all* of the near-extremal cases of current interest.

In summary, we have been discussing the effect of thermal fluctuations on the entropy and stability of a black hole. More specifically, we have reviewed a prior canonical/grand canonical treatment [1] that is novel for (at least) two reasons. Firstly, our methodology emphasizes the distinction between systems with a fixed charge and those for which the charge is dynamical. Secondly, our formalism directly incorporates the black hole area spectrum, which should be, as we have argued [1], regarded as an essential ingredient in any calculation of this nature. Following this approach, we were able to explore the thermodynamic behavior of black holes when near certain critical points. For instance, it was shown that, due to thermal fluctuations, instability sets in well before the system reaches either one of

⁶As one might imagine, there are additional complications that arise when the charge is fluctuating; *cf.* case *iii*. Here, it is useful to fix $p = 1$, as is appropriate for a small L regime. Then it follows that, close to extremality, $\mathcal{A}_0 \sim m_0$, so that both \mathcal{A}_0 and m_0 can be expanded in terms of $|m_0| = |m_c| - \varepsilon$, where m_c is the critical (extremal) value. Also note that, for the purpose of assessing stability, both Δn and Δm need to be compared with ε .

the *phase-transition* points; that is, the Hawking-Page phase transition [16] or the transition point of Reissner-Nordstrom black holes [17]. More precisely, in both of these cases, instability will occur at the order of $A^{2/3}$ away from the critical value of A . On the other hand, the thermal fluctuations are completely suppressed for a system close to extremality. Therefore, in a near-extremal regime, instability ensues at only a few Planck scales below the extremal point.

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